GROUP TEST: ALGEBRA

(1) (a) The group SO(2) acts on $\mathbb{C}[X,Y]$ as follows:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot P(X,Y) = P(\cos\theta X + \sin\theta Y, -\sin\theta X + \cos\theta Y).$$

Prove that its invariant subspace $\mathbb{C}[X,Y]^{\mathrm{SO}(2)} = \mathbb{C}[X^2+Y^2]$.

(b) Let $\mathcal{S} = \mathbb{C}[X,Y]e^{-\pi(X^2+Y^2)}$ and let F be the Fourier transformation:

$$F\phi(x,y) = \int_{\mathbb{R}^2} \phi(u,v)e^{2\pi i(ux+vy)}dudv.$$

Prove that for any $P \in \mathbb{C}[X,Y], \phi \in S$,

$$F(P \cdot \phi) = P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) F(\phi).$$

(c) Describe the space $\mathbb{C}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]^{\mathrm{SO}(2)}$.

(2) (a) What is the Galois group of $\mathbb{Q}(\zeta_7)$ over $\mathbb{Q}$, where $\zeta_7 = e^{2\pi i/7}$?

(b) Find a Galois extension field F of $\mathbb{Q}$ such that $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$? Are there infinitely many such Galois extensions?

(c) Find the quadratic subfield of $\mathbb{Q}(\zeta_7)$?